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A Lagrangian solution for internal waves

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A perturbation procedure is used to obtain first- and second-order solutions for small-amplitude internal waves in a Lagrangian coordinate system. The first-order Lagrangian equations are formally accurate to the same order as the first-order Eulerian equations; however, they are different and the Lagrangian solution gives a more realistic wave shape. First-order Lagrangian solutions for internal waves in uniformly stratified fluid have a shape similar to that found in the second-order Eulerian solution. Wave profiles in uniformly stratified fluid exhibit broad crests and narrow troughs near the surface, a sinusoidal shape at mid-depth, and narrow crests and broad troughs near the bottom. The difference between the shape of crests and troughs grows as the wave amplitude is increased. Solutions obtained in a uniformly stratified fluid with a small bottom slope yield plausible shapes for breaking waves.

1. Introduction

Comparatively little effort has been made to model oceanographic phenomena in the Lagrangian coordinate system compared with that expended on Eulerian equations. This is perhaps understandable in view of the degree of nonlinearity of the complete Lagrangian equations of motion, compared with that of the Eulerian equations of motion (Pierson 1962). In particular, the pressure gradient force is nonlinear in Lagrangian coordinates, so when the flow is quasi-geostrophic the advantage obviously lies with the Eulerian formulation. Further, the Lagrangian form of the equation of continuity is nonlinear, and this equation generally is not regarded as one to be trifled with, although Pierson (1962) justifies the linearized form by referring to work done on surface gravity waves by Miche (1944). Other problems can arise with some types of boundary conditions where particles originally at the boundary are advected away from the boundaries. Such boundary conditions are often trivial in Eulerian coordinates but become complicated functions of space and time in Lagrangian coordinates. This results in a situation where it virtually becomes necessary to know the solution in order to specify the boundary conditions. However, for many situations (for example when there is no motion at the boundaries) the Lagrangian form of the boundary conditions is as tractable as the corresponding Eulerian form.

With so much running against the Lagrangian formulation one might well ask if it has any advantage at all. One advantage is that the total acceleration is linear in the Lagrangian formulation whereas it is decidedly nonlinear in the Eulerian formulation. This leads us to suspect that the Lagrangian formulation may be useful for describing phenomena which have a small enough scale so that the Eulerian field accelerations are important. This point will be clearly illustrated in the following work, where a first-order solution for internal waves will be found in Lagrangian coordinates.

The usefulness of the Lagrangian formulation has been previously illustrated by Pierson (1962) who found solutions that looked somewhat like turbulence, and by Okubo (1967) who used such solutions along with the Lagrangian diffusion equation in order to describe eddy diffusion. Both of these studies considered homogeneous fluids. The Lagrangian solution has also been applied with some success to surface waves (Miche 1944; Biesel 1952; Neumann & Pierson 1966). In particular, some features of the shape of surface waves are described just as well with the first-order solution of the Lagrangian equations as with the third-order solution of the Eulerian equations, Neumann & Pierson (1966). With this in mind Neumann & Pierson (1966) suggest that it might be advantageous to find a Lagrangian solution in order to describe the shape of internal waves. In particular, the shape of internal waves at different depths in linearly stratified uniform-depth fluid will be treated. The shape of internal waves in two-layer and continuously stratified fluids has been studied previously by Thorpe (1968), Hunt (1961) and Orlanski (1972), among others, by considering up to third-order solutions in Eulerian coordinates. Olbers (1976) has used a Lagrangian approach to investigate nonlinear energy transfer between components of the internal wave spectrum.

Realistic-looking Lagrangian solutions for surface gravity waves propagating up a slope have been found by Biesel (1952). This suggests that it might also be profitable to study breaking internal waves as they propagate in a uniformly stratified ocean of variable depth.

The shape of internal waves is inherently Lagrangian since the fundamental variable is displacement of the isopycnals (not velocity). A similar observation led Okubo (1967) usefully to employ Lagrangian coordinates to study eddy diffusion.

2. Formulating the problem in Lagrangian coordinates

The two-dimensional nonviscous form for the Lagrangian equations of motion in a nonhomogeneous fluid are given by

$$x_{tt}\,x_a+(z_{tt}+g)\,z_a+\frac{P_a}{\rho}=0, \tag{1}$$

$$x_{tt}\,x_c+(z_{tt}+g)\,z_c+\frac{P_c}{\rho}=0. \tag{2}$$

Neumann & Pierson (1966), page 122, and Lamb (1932) show how these equations can be derived from the equivalent Eulerian equations by transforming the variables. The variables x and z are physically the particle coordinates. Pressure is represented by P and density by ρ. The dependent variables x, z, P are all functions of time and the initial particle coordinates (a, c).

In order to close the system we introduce the equation of mass conservation

$$\rho(a,c,t)\,\frac{\partial(x,z)}{\partial(a,c)}=\rho(a,c,0). \tag{3}$$

Assuming that the fluid is incompressible,

$$\rho(a,c,t)=\rho(a,c,0), \tag{4}$$

then (3) reduces to

$$\frac{\partial(x,z)}{\partial(a,c)}=1. \tag{5}$$

If the fluid is incompressible then (1) and (2) can be rearranged into the following form:

$$x_{tt}+\frac{P_a z_c - P_c z_a}{\rho} = 0, \tag{6}$$

$$z_{tt}+\frac{P_c x_a - P_a x_c}{\rho}+g = 0, \tag{7}$$

by invoking (5) and by using (2) to eliminate z_{tt} from (1) in order to obtain (6). Similarly (1) can be employed to eliminate x_{tt} from (2) in order to obtain (7). The equations are more physically understandable in this form. The horizontal and vertical components of the particle acceleration are given by x_{tt} and z_{tt} respectively. The pressure gradient force is nonlinear and takes the form of products of Lagrangian deformations and gradients in the pressure. Clearly if the Lagrangian displacements are very small then $z_c \approx x_a \approx 1$ and $z_a \approx x_c \approx 0$, in which case the equations assume a simple linear form.

Equations (1), (2) and (5) are nonlinear and need to be linearized. Assume that x, z, P can be expanded as a perturbation series about $x = a$, $z = c$, and $P = P_0$:

$$x = a+\epsilon x_1+\epsilon^2 x_2+\dots, \tag{8}$$

$$z = c+\epsilon z_1+\epsilon^2 z_2+\dots, \tag{9}$$

$$P = P_0+\epsilon P_1+\epsilon^2 P_2+\dots. \tag{10}$$

Note that a slightly different expansion will be required if there is a mean shear (that is a shear that is not directly caused by the internal waves). The internal wave interaction with a mean vertical shear is currently under investigation. In the above expansions ϵ is an ordering parameter which will be subsequently set equal to 1. It will also be convenient to consider the density as being composed of zeroth- and first-order components

$$\rho = \rho_0+\epsilon\rho_1, \tag{11}$$

where both ρ_0 and ρ_1 are independent of time, being equal to the specified initial values which are functions of space. Pierson (1962) used a similar perturbation expansion for (8), (9) and (10) for a homogeneous fluid. The assumption implicit in the perturbation expansion is that the Lagrangian deformations are small. This is equivalent to assuming that the wave amplitude is small compared to the vertical scale of the wave. The convergence of the series will be discussed later when first- and second-order solutions have been found.

Substituting (8), (9), (10), (11) into (1), (2), (5) yields the following zeroth-order equations:

$$P_{0a} = 0, \tag{12}$$

$$P_{0c}+\rho_0 g = 0 \tag{13}$$

and the zeroth-order equation of continuity is satisfied exactly. From (12) and (13) we see that ρ_0 is a function of c alone, which is why the first-order term is needed in (11) if we are to consider density to also be a function of a. Note that if density is a function of a then this will cause horizontal pressure gradients which will result in some mean motion. In the present formulation the only force available to act against a horizontal pressure gradient is that due to the inertial acceleration. Such a situation would be quite unrealistic for the ocean and so ρ_1 is set equal to zero throughout the following work. However if Coriolis accelerations were included in the formulation then it would be realistic to consider horizontal variations of the density field, as was done by Mooers (1975) in an Eulerian coordinate system.

In the same fashion the first-order equations are

$$x_{1tt}+gz_{1a}+\frac{P_{1a}}{\rho_0}=0, \tag{14}$$

$$z_{1tt}+gz_{1c}+\frac{P_{1c}}{\rho_0}=0, \tag{15}$$

$$x_{1a}+z_{1c}=0. \tag{16}$$

Notice that the linearized continuity equation, (16), has a different form from its true nonlinear form (5); hence the first-order solutions will not be expected to satisfy continuity exactly. On the other hand (4) is satisfied exactly by the first-order solutions whereas the Eulerian equivalent of (4) is $\rho_t+u_x\rho+w_z\rho=0$, which is only approximately satisfied by the solution of the linearized Eulerian equations. For both the Lagrangian and Eulerian equations the order of the linearizing approximation is the same, although the approximation is different.

It becomes clear that the first-order pressure terms of (14) and (15) are of a different form to those of the nonlinear (6) and (7). However the particle acceleration term of the first-order equations is of exactly the same form as in the nonlinearized equations. Thus the first-order Lagrangian equations retain field acceleration terms (which are of second order) at the expense of introducing second-order errors into the pressure gradient terms. The corresponding first-order Eulerian equations neglect field acceleration terms but preserve the form of the pressure gradient terms.

Formally, therefore, the first-order Eulerian equations and the first-order Lagrangian equations are equally accurate. However the first-order Lagrangian equations are not the same as the first-order Eulerian equations since they represent different physical approximations to the unlinearized equations. Further, we note that the first-order Lagrangian equations have a nonlinear form when transformed into Eulerian coordinates. It seems likely, therefore, that when the solution to the first-order Lagrangian equations is transformed back into Eulerian coordinates a nonlinear-looking wave might result. With luck some of these nonlinear features might correspond to observed features that are caused by the field acceleration terms. However, meaningless nonlinear features might equally well result from the approximate treatment of the pressure gradient term.

Eliminating the pressure from the first-order equations by cross-differentiation yields

$$\rho_0 x_{1ttc}-\rho_0 z_{1tta}+\rho_{0c}x_{1tt}+\rho_{0c}gz_{1a}=0. \tag{17}$$

By the Boussinesq approximation the term $\rho_{0c}x_{1tt}$ is generally small enough to be neglected for internal waves found in the ocean (Groen 1948).

Equation (16) permits the introduction of a stream function, $\psi(a,c)$, such that

$$z_1=-\psi_a\,\mathrm{e}^{-\mathrm{i}\omega t}, \tag{18}$$

$$x_1=\psi_c\,\mathrm{e}^{-\mathrm{i}\omega t}, \tag{19}$$

which upon substitution into (17) yields

$$\psi_{cc}-\left[\frac{N^2-\omega^2}{\omega^2}\right]\psi_{aa}=0, \tag{20}$$

where $N=(-g\rho_{0c}/\rho_0)^{\frac{1}{2}}$ is the Brunt–Väisälä frequency which will be taken to be a constant throughout the following work. The concept of the stream function $\psi(a, c)$ in Lagrangian coordinates is essentially the same as in Eulerian coordinates, in that lines of constant ψ represent a curve whose tangent is everywhere parallel to the local instantaneous particle displacement. We note however that a stream function cannot be defined for the complete nonlinear problem, see (1), (2) and (5), whereas a stream function can be defined for the equivalent nonlinear problem expressed in Eulerian coordinates.

3. Solutions for a flat bottom

If the fluid is considered to have a uniform depth h, then the boundary conditions

$$\psi(a, 0) = 0, \tag{21}$$

$$\psi(a, -h) = 0 \tag{22}$$

are imposed. Equation (21) states that the surface is a rigid lid. Phillips (1966, §5.2) shows that this approximation is usually valid. Equation (22) states that the ocean's bottom is also a streamline, so that there is no motion across the bottom. Equations (21) and (22) together imply no mean horizontal flow.

The problem is observed to have exactly the same mathematical form as the equivalent Eulerian formulation, and therefore has a solution, for progressive waves, of the form

$$\psi(a, c) = \mathrm{i}\,\frac{A}{k_n}\sin\left(\frac{n\pi c}{h}\right)\exp\left(\mathrm{i}k_n a\right), \tag{23}$$

where

$$k_n = \frac{n\pi\lambda}{h}, \quad n = 1, 2, 3, \ldots \tag{24}$$

and

$$\lambda^2 = \frac{\omega^2}{N^2-\omega^2}, \quad \omega < N. \tag{25}$$

In the above expressions A is the vertical amplitude coefficient (units m) and the horizontal wavenumber for the nth mode is given by k_n (units m^{-1}). To first order the particle displacements are given by

$$x = a - \frac{A}{\lambda}\cos\left(\frac{n\pi c}{h}\right)\sin\left(k_n a - \omega t\right), \tag{26}$$

$$z = c + A\sin\left(\frac{n\pi c}{h}\right)\cos\left(k_n a - \omega t\right), \tag{27}$$

which are of a similar form to expressions for the velocity found from the first-order Eulerian equations. The variables are not, however, the same as for the Eulerian solution, reflecting the different physical approximation to the complete nonlinear equations. In order to observe the shape of the wave we plot the isopycnals for a chosen depth c (the depth of an isopycnal without internal waves) and time t by plotting x against z as a is varied parametrically. This is valid because the density of particles is not a function of time. The same technique has previously been employed by Neumann & Pierson (1966) and Biesel (1952), in order to plot the shape of surface gravity waves. Note that the solution given by (26) and (27) also defines the transformation between Lagrangian and Eulerian coordinates. The transformation will be accurate to the same order as the solution. It is possible to manipulate (26)

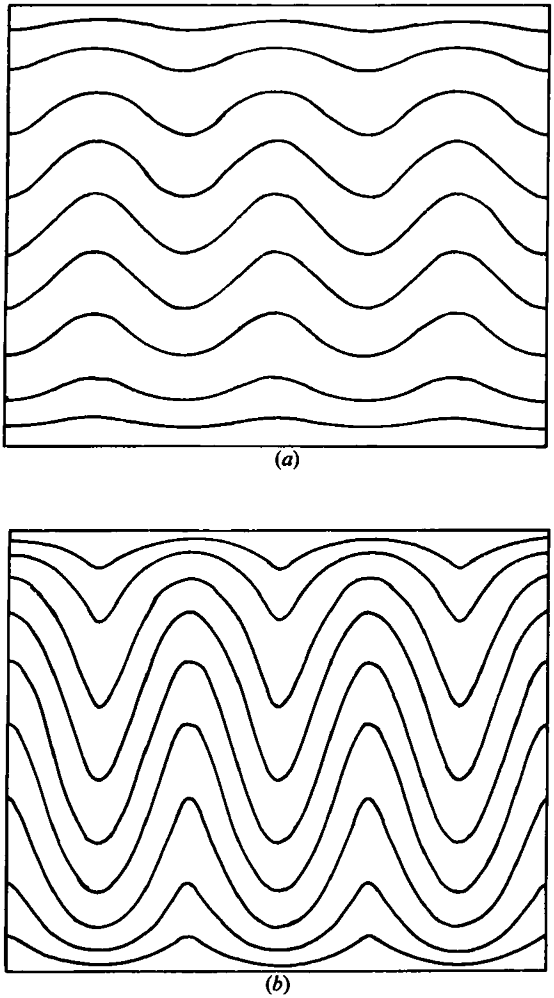

FIGURE 1. (a) The shape of the first-order solution for the first-mode internal wave at depths of 10, 25, 50, 75, 100, 125, 150, 175, and 190 m. The wavelength is 916.5 m, the water depth is 200 m, and the vertical amplitude A is 13.7 m. The ratio of the vertical to horizontal scale is 10 to 1. (b) The shape of a wave with the same parameters as in figure 1(a) but with a larger amplitude $A = 41.1$ m. The ratio of the vertical to horizontal scale is 10 to 1.

and (27) to obtain an expression for x as a function of (z, c, t) but not, unfortunately, to obtain z as a function of (x, c, t). This is why a parametric approach has been used to plot wave shapes.

Plots of wave shape for the first mode, are shown in figures 1(a) and 1(b) for two different amplitudes and various depths. Waves in both of these plots have a sufficiently large Richardson number to be considered stable. Clearly the shape of the waves is now a function of both wave depth and amplitude. The waves have a marked nonsinusoidal shape even though x and z are sinusoidal functions of a and c. Clearly, this is caused by the transformation between Eulerian and Lagrangian coordinates (as expressed to first order by (26) and (27)) being nonlinear. Near the surface wave peaks are broad and wave troughs are narrow. At mid-depth the troughs and crests have the same shape. Near the bottom the wave troughs are broad and the wave crests are peaked. As the amplitude of the waves becomes greater we observe that the waves become increasingly peaky.

The particle trajectory can be found (to first order) by eliminating a from expressions for x and z. The resulting expression is the equation for an ellipse:

$$\left[\frac{\lambda x_1}{\cos\left(\frac{n\pi c}{h}\right)}\right]^2+\left[\frac{z_1}{\sin\left(\frac{n\pi c}{h}\right)}\right]^2 = A^2. \tag{28}$$

For the first mode the ellipse has ellipticity λ at mid-depth and flattens to straight horizontal lines at the surface and bottom.

Let us now compare the wave shape given by the first-order Lagrangian solution with that given by the first- and second-order Eulerian solutions of Thorpe (1968). These three wave shapes are compared in figure 2. Clearly the first-order Lagrangian solution is superior to the first-order Eulerian solution for describing the shape of the internal wave.

Even more realistic wave shapes might be anticipated if the problem is solved to second order. The second-order perturbation equations are

$$x_{2tt}+gz_{2a}+\frac{P_{2a}}{\rho_0} = -x_{1tt}\,x_{1a}-z_{1tt}\,z_{1a}, \tag{29}$$

$$z_{2tt}+gz_{2c}+\frac{P_{2c}}{\rho_0} = -x_{1tt}\,x_{1c}-z_{1tt}\,z_{1c}, \tag{30}$$

$$x_{2a}+z_{2c} = x_{1c}\,z_{1a}-x_{1a}\,z_{1c}. \tag{31}$$

Substituting expressions for the first-order terms into the right-hand side of the above equations and using the method of undetermined coefficients it is fairly straightforward to find the following solutions for the second-order terms:

$$x_2 = \frac{-\mathrm{i}k_n A^2}{2\lambda^2}\,\mathrm{e}^{\mathrm{i}2(k_n a-\omega t)}, \tag{32}$$

$$z_2 = 0, \tag{33}$$

$$P_2 = \frac{\omega^2\rho_0 A^2}{\lambda^2}\left[\frac{\lambda^2}{2}\sin^2\left(\frac{n\pi c}{h}\right)-\tfrac{1}{2}\cos^2\left(\frac{n\pi c}{h}\right)-1\right]\mathrm{e}^{\mathrm{i}2(k_n a-\omega t)}. \tag{34}$$

These second-order corrections have the effect of making the sides of the waves steeper while also making the troughs and crests broader, as is shown in figure 3, which is the second-order solution for the same wave plotted in figure 1 (*b*). The second-order solution compares very favourably with wave shapes measured in the ocean (LaFond 1962). The second-order Lagrangian solution is compared with the second-order Eulerian solution in figure 4. Again the Lagrangian solution has a broader trough and rises more steeply than the Eulerian solution.

The first- and second-order stream functions are identical in Eulerian coordinates and the first-order Lagrangian solution for the stream function is different, as shown in figure 5. We expect the second-order Eulerian solution for the stream function to be more accurate than the first-order Lagrangian solution. Therefore, it would appear that the velocity field is best described using Eulerian equations. Shear instability criteria are therefore best found from Eulerian solutions. The first-order Lagrangian solution predicts that gravitational instability, resulting from particle speeds becoming greater than the phase speed, will occur only for waves with such large amplitudes

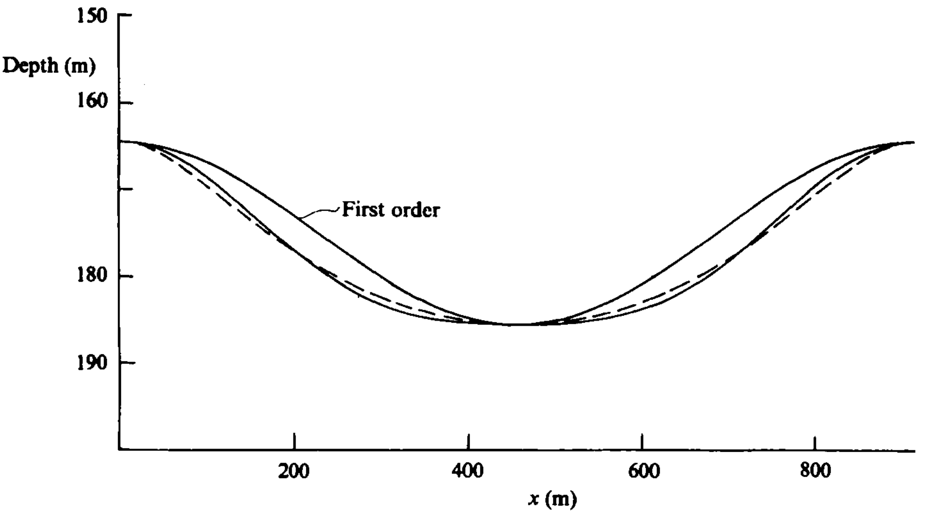

FIGURE 2. A comparison of the wave shape for first- and second-order Eulerian solutions (——) with that for the first-order Lagrangian solution (---). The wavelength is 916.5 m, the water depth is 200 m, the depth of the isopycnal plotted is 175 m, and the vertical amplitude coefficient A is 27.4 m. The ratio of vertical to horizontal scales is 10 to 1.

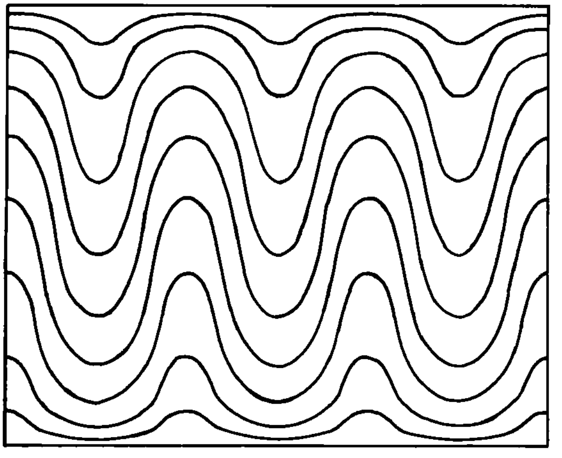

FIGURE 3. The shape of the wave plotted in figure 1 (*b*) with second-order corrections added.

that they would break the surface. For the present solution this type of instability is obviously unimportant (as it also is for the Eulerian case). Gravitational instability could, however, occur if the internal wave interacted with a mean vertical shear.

For the perturbation procedure to be valid it is necessary for higher-order terms in the series solution to become smaller. The second-order terms are smaller than the first-order terms by a factor of $n\pi A/2h$, which gives us some confidence that the series converges for waves that have a small vertical amplitude A compared with the vertical scale of the wave, h/n.

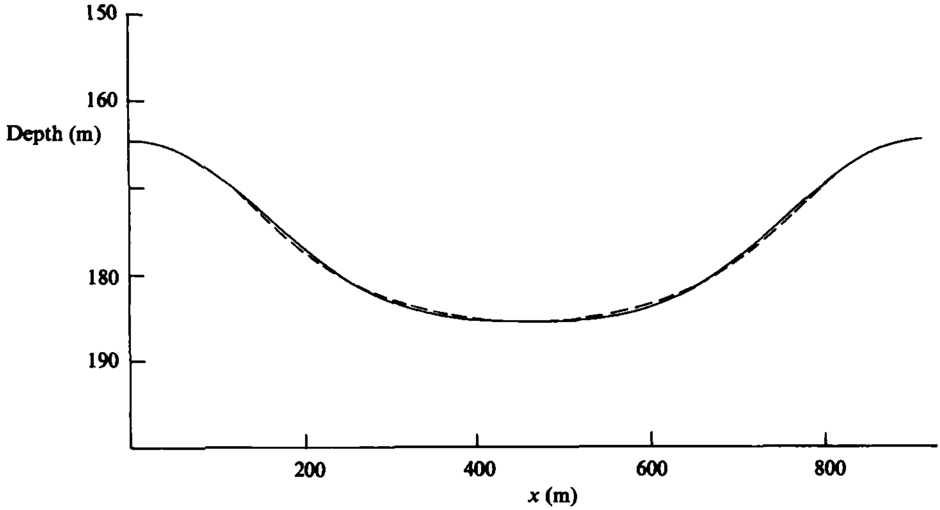

FIGURE 4. A comparison of the second-order Eulerian wave shape (——) with the second-order Lagrangian wave shape (---). The parameters are the same as for the wave in figure 2.

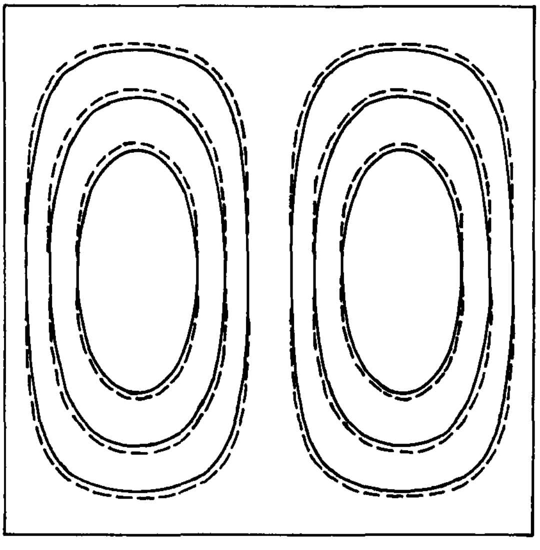

FIGURE 5. A comparison of first- and second-order Eulerian streamlines (——) with first-order Lagrangian streamlines (---) for a wave with the same characteristics as that plotted in figure 2. The first- and second-order streamlines are identical in Eulerian coordinates. The ratio of the vertical scale to horizontal scale is 4.6 to 1.

4. Solutions for other depth profiles

If the bottom of the ocean has a uniform slope s then to first order the problem is defined by (20), (21) and

$$\psi(a, -h) = 0, \tag{35}$$

where to first order we can write

$$h = sa + h_0, \tag{36}$$

if the bottom slope is small, $s = O(\epsilon)$. These equations have the same form as those of the corresponding Eulerian problem which was treated by Manton (1970), Manton

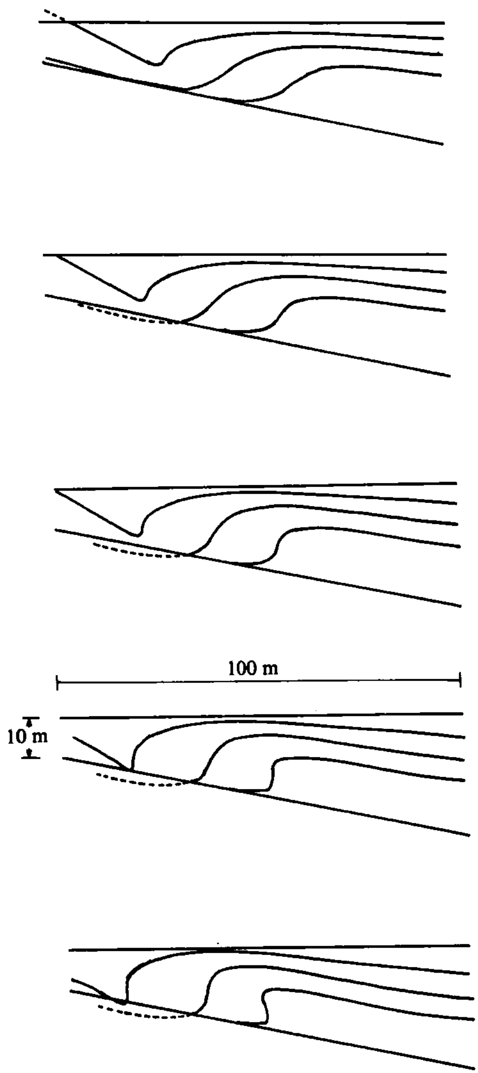

FIGURE 6. The evolving shape of an internal wave at 200 s intervals as it propagates up a uniformly sloping sea bed. Parameters are $s = 0.2$, $N = 0.005$ rad/s, $\omega = 0.002$ rad/s, $A' = 40$, and the unperturbed depths of the isopycnals shown are 5, 10, and 15 m.

& Mysak (1971), and Wunsch (1968, 1969). From this work we see that a solution for a uniformly sloping bottom is

$$\psi(a,c) = A' \exp\left[\mathrm{i}2n\pi R \ln\left(a+\frac{c}{\lambda}+\frac{h_0}{s}\right)\right] - A' \exp\left[\mathrm{i}2n\pi R \ln\left(a-\frac{c}{\lambda}+\frac{h_0}{s}\right)\right], \quad (37)$$

where
$$R = \frac{1}{\ln\left[\dfrac{1+s/\lambda}{1-s/\lambda}\right]}, \quad (38)$$

and $n = 1, 2, 3 \ldots$. The shape of these waves can be found by plotting x and z (for given depths c and times t) as parametric functions of a. Figure 6 shows the shape of a progressive wave at 200 s intervals as it runs up the sloping bottom. The isopycnals, at depths of 5, 10, and 15 m for the unperturbed fluid, are plotted.

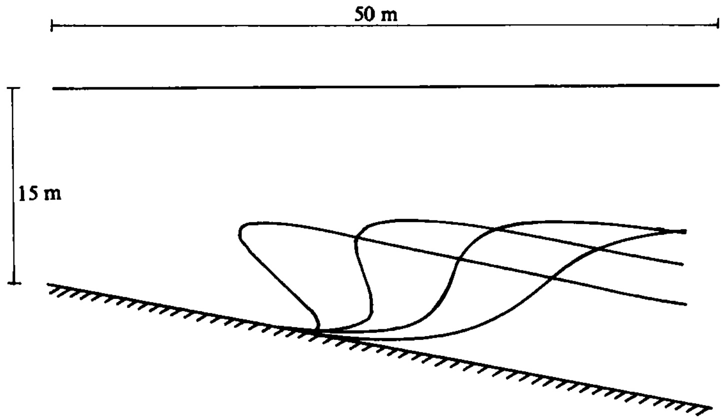

FIGURE 7. Detail of the changing shape of the 15 m isopycnal at 400 s intervals as the internal wave plotted in figure 6 propagates up a slope.

Clearly, near the surface the wave breaks downwards whereas at depth it breaks upwards. The bottom of the wave breaks closer to shore than the top of the wave. The wave steepens as time progresses, becomes very asymmetric and even becomes double-valued. The wave is somewhat unrealistic since it goes beneath the bottom (a consequence of continuity not being satisfied exactly). Nevertheless the solution certainly has a plausible shape, for a first approximation. More realistic shapes near breaking would probably require the addition of frictional forces since there are strong shears at this point. Friction might be expected to reduce the amplitude of the waves as they break, thereby avoiding the situation shown in the last plot of figure 6, where a wave trough is breaking through the sea floor. The evolution of the wave at 15 m depth is shown at 400 s time steps in figure 7. Clearly the wave becomes double-valued and therefore gravitationally unstable, so that if kinetic energy was lost to friction then mixing would occur.

5. Conclusions

The present work illustrates how some features of internal waves can be described more easily in Lagrangian than in Eulerian coordinate systems. In particular the shape of internal waves, propagating in an ocean of uniform stratification and constant depth, shows the observed broadening of the crests/troughs near the surface/bottom as well as the generally observed steep sides of the wave (compared with the sinusoidal shape). The solution for a wave propagating up a small slope appears to yield a fairly realistic shape for the initial stages of wave breaking. A variety of solutions for other bottom profiles could be obtained easily by using existing solutions for the Eulerian formulation, such as those of Manton (1970).

Throughout the above work we see that the form of the first-order equations is similar in both Eulerian and Lagrangian coordinate systems. It appears possible that other problems that use linearized Eulerian equations could be profitably examined in a similar fashion.

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